UNIQUENESS AND BOUNDARY BEHAVIOUR OF LARGE SOLUTIONS TO ELLIPTIC PROBLEMS WITH SINGULAR WEIGHTS *

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Abstract

We consider the elliptic problems $\Delta u = a(x)u^m$, $m > 1$, and $\Delta u = a(x)e^u$ in a smooth bounded domain Ω, with the boundary condition $u = +\infty$ on $\partial\Omega$. The weight function $a(x)$ is assumed to be Hölder continuous, growing like a negative power of $d(x) = \operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$. We show existence and nonexistence results, uniqueness and asymptotic estimates near the boundary for both the solutions and their normal derivatives.

1. Introduction

Elliptic problems with blow-up on the boundary have been intensively studied in the past few years. One usually looks for a function $u \in C^2(\Omega)$ solving an elliptic equation in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and such that $u(x) \to +\infty$ as $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0+$.

In this paper we study the problems corresponding to nonlinearities of power and of exponential type. More precisely,

$$\begin{cases} \Delta u = a(x)u^m & \text{in} \quad \Omega \\ u = +\infty & \text{on} \quad \partial\Omega \end{cases} \tag{P}$$

and

$$\begin{cases} \Delta v = a(x)e^v & \text{in} \quad \Omega \\ v = +\infty & \text{on} \quad \partial\Omega, \end{cases} \tag{P'}$$

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where $a(x)$ is a weight function.

Problem (P) was studied for the first time in [23], with $a \equiv 1$ and $p = (N+2)/(N-2)$. Later, in [20], [2], [3] and [27], the case with $a(x)$ smooth and positive up to the boundary and, in [14], the case where a is smooth but vanishes on the boundary were considered. The extension to p-Laplacian equations is studied in [9]. In all these papers, uniqueness is shown by means of an estimate of the form $u \sim Cd^{-\alpha}$ as $d \to 0+$. We also quote [19], where uniqueness with $a \equiv 1$ is obtained without exact estimates for solutions on the boundary.

As for problem (P'), it was considered for the first time in [5] ($a \equiv 1$), and later in [21] (a smooth and positive up to $\partial\Omega$) and [1] ($a \equiv 1$). In [21] and [1], uniqueness and asymptotic estimates near the boundary were also provided.

More recently, some results have appeared which treat the situation where the weight a is possibly unbounded near $\partial\Omega$. In [25], the radial case was considered for more general operators than the Laplacian, but obtaining only existence and nonexistence results. For problem (P) (including the range $0 < m \le 1$), the radial case was completely discussed in [6]; the authors obtained existence, nonexistence, uniqueness, multiplicity and estimates for all positive solutions. Finally, in [28] and [13], the existence of solutions in smooth bounded domains of $\mathbb{R}^N$ was shown for both problems (P) and (P') (actually [13] admits more general power-like and exponential-like nonlinearities without monotonicity assumptions).

We mention in passing other relevant papers where related problems were considered, for instance [4], [8], [12], [18], [19], [22], [24] and [26] (see also [7], [10], [11], [15], [16] for elliptic systems).

Our goal here is to provide a complete picture of the set of solutions to problems (P) and (P') when the weight $a(x)$ is singular on $\partial\Omega$. We show existence, nonexistence, uniqueness and estimates near the boundary for solutions and their normal derivatives in smooth bounded domains of $\mathbb{R}^N$. Techniques based on sub and supersolutions were used in [8], [14] and [16] . We will proceed in a different manner, which in particular will give us information about the growth of the normal derivatives of the solutions. For the sake of completeness we also prove existence, although by a different method than that in [28] and [13].

We will assume that Ω is a $C^{2,\mu}$ bounded domain for some $0 < \mu < 1$, and that the weight $a(x)$ is a locally μ–Hölder continuous function which verifies

$$C_1 d(x)^{-\gamma} \le a(x) \le C_2 d(x)^{-\gamma}, \quad x \in \Omega \tag{A}$$

for some positive constants γ, C_1, C_2. Under these assumptions, we will show that problems (P) and (P') only admit solutions when $0 < \gamma < 2$, in which case they are unique and satisfy bounds in terms of the data. For certain estimates near the boundary, we will sometimes need to assume the stronger condition:

$$\left\{ \begin{array}{l} \text{there exists a bounded and positive function } C_0 \text{ defined on } \partial\Omega \\ \text{such that } \lim_{x\to x_0} d(x)^\gamma a(x) = C_0(x_0) \text{ for every } x_0 \in \partial\Omega. \end{array} \right. \tag{A'}$$

We will see that this implies estimates of the growth of both the solutions and their normal derivatives near $\partial\Omega$. Note that problem (P) only makes sense for positive solutions (unless we define u^m in a suitable way for negative values of u), while the solutions in (P') may change sign. Our results are the following.

Theorem 1. *Assume $a \in C^{\mu}(\Omega)$ verifies hypotheses (A). Then problem (P) has no positive solutions if $\gamma \geq 2$, and it has a unique positive solution $u \in C^{2,\mu}(\Omega)$ when $0 < \gamma < 2$. Moreover,*

$$(D_1 \sup_{\Omega} d(x)^{\gamma} a(x))^{-\frac{1}{m-1}} d(x)^{-\alpha} \leq u(x) \leq (D_2 \inf_{\Omega} d(x)^{\gamma} a(x))^{-\frac{1}{m-1}} d(x)^{-\alpha} \quad \text{in } \Omega, \tag{1}$$

where $\alpha = (2-\gamma)/(m-1)$, and D_1, D_2 are positive constants depending only on α and Ω. If in addition a verifies condition (A'), then

$$\begin{gathered}\lim_{x \to x_0} d(x)^{\alpha} u(x) = \left(\frac{\alpha(\alpha+1)}{C_0(x_0)}\right)^{\frac{1}{m-1}}, \quad \lim_{x \to x_0} d(x)^{\alpha+1} \nabla u(x) \nu(x_0) = \alpha \left(\frac{\alpha(\alpha+1)}{C_0(x_0)}\right)^{\frac{1}{m-1}}, \\ \lim_{x \to x_0} d(x)^{\alpha+2} \text{Hess } u(x)[\nu(x_0), \nu(x_0)] = \alpha(\alpha+1) \left(\frac{\alpha(\alpha+1)}{C_0(x_0)}\right)^{\frac{1}{m-1}},\end{gathered} \tag{2}$$

for every $x_0 \in \partial\Omega$, where Hess u *denotes the Hessian of u and ν stands for the exterior unit normal to $\partial\Omega$.*

Theorem 2. *Assume $a \in C^{\mu}(\Omega)$ verifies hypotheses (A). Then problem (P') has no solutions if $\gamma \geq 2$, and it has a unique solution $v \in C^{2,\mu}(\Omega)$ when $0 < \gamma < 2$. Moreover,*

$$D_1'(2-\gamma)(\sup_{\Omega} d(x)^{\gamma} a(x))^{-1} d(x)^{\gamma-2} \leq e^{v(x)} \leq D_2'(2-\gamma)(\inf_{\Omega} d(x)^{\gamma} a(x))^{-1} d(x)^{\gamma-2} \quad \text{in } \Omega, \tag{3}$$

where D_1', D_2' are positive constants depending only on Ω. If in addition a verifies the asymptotic condition (A'), then

$$\begin{gathered}\lim_{x \to x_0} (v(x) + (2-\gamma)\log d(x)) = \log\left(\frac{2-\gamma}{C_0(x_0)}\right), \quad \lim_{x \to x_0} d(x) \nabla v(x) \nu(x_0) = (2-\gamma), \\ \lim_{x \to x_0} d(x)^2 \text{Hess } u(x)[\nu(x_0), \nu(x_0)] = (2-\gamma)\end{gathered} \tag{4}$$

for every $x_0 \in \partial\Omega$, where ν stands for the exterior unit normal to $\partial\Omega$.

Remarks 1. a)Existence of solutions holds under the weaker assumption $0 < a(x) \leq C_2 d(x)^{-\gamma}$ in Ω, for some $0 < \gamma < 2$, $C_2 > 0$.

b) Estimate (3) implies the weaker estimate

$$\lim_{x \to x_0} \frac{v(x)}{-\log d(x)} = 2 - \gamma \tag{5}$$

for every $x_0 \in \partial\Omega$. Condition (A') is not needed for (5) to hold.

c) As a byproduct of the proof of estimates (2) and (4) (see Section 4) it follows that the tangential derivatives of the solutions grow at a lower rate than the normal derivatives.

d) If the weight $a(x)$ is more regular, say $a \in C^{k,\mu}(\Omega)$ and Ω is $C^{k+2,\mu}$, for some $k \in \mathbb{N}$, then we can establish estimates similar to (2) and (4) for the derivatives up to order $k+2$ in the direction of the normal.

The paper is organized as follows: Section 2 is devoted to the existence of solutions to problems (P) and (P'). In Section 3 we obtain estimates (1) and (3) which, in particular, imply uniqueness and nonexistence when $\gamma \geq 2$. Finally, Section 4 deals with the proof of the estimates (2) and (4). Theorems 1 and 2 will then follow.

2. Existence of solutions

Existence of solutions of problems (P) and (P') is already known from [28] and [13] when $0 < \gamma < 2$. We give here a different proof based on a standard technique. We construct solutions with finite datum $n \in \mathbb{N}$ and then obtain the desired solution by passing to the limit as $n \to +\infty$. The key step in this process is to show local uniform bounds.

Lemma 3. *Let $n \in \mathbb{N}$, and assume $a \in C^\mu(\Omega)$ verifies (A) with $0 < \gamma < 2$. Then the problem*

$$\begin{cases} \Delta u = a(x)u^m & in \quad \Omega \\ u = n & on \quad \partial\Omega \end{cases} \tag{P_n}$$

admits a unique positive solution $u_n \in C^{2,\mu}(\Omega) \cap C(\overline{\Omega})$. Moreover, u_n is increasing in n.

Proof. Choose a cut-off function $\varphi \in C^1(\mathbb{R}^+)$ such that $0 \le \varphi \le 1$, $\varphi(t) = 0$ if $0 \le t \le 1$, $\varphi(t) = 1$ for $t \ge 2$, and for $k \in \mathbb{N}$ set $a_k(x) = a(x)\varphi(kd(x))$. Then $a_k \in C^\mu(\overline{\Omega})$ (we recall that $d(x)$ is always Lipschitz-continuous). Consider the problem

$$\begin{cases} \Delta w = a_k(x)(w+n)^m & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega\ . \end{cases} \tag{6}$$

Since we can take $\underline{w} = -n$, $\bar{w} = 0$ as ordered sub and supersolutions, respectively, the existence of at least a solution $w_k^n \in C^{2,\mu}(\overline{\Omega})$ is established. By the monotonicity of the right-hand side and the maximum principle it follows that w_k^n is unique.

In order to pass to the limit as $k \to +\infty$ in (6)we need the following lemma which appears in [17] (see Lemma 4.9 and Problem 4.6).

Lemma 4. *Let Ω be a C^2 bounded domain of $\mathbb{R}^N$ and $f \in C^\mu(\Omega)$ such that $\sup_\Omega d(x)^\gamma |f(x)| < +\infty$ for some $1 < \gamma < 2$. Then the problem $\Delta u = f$ in Ω with $u = 0$ on $\partial\Omega$ has a unique solution $u \in C^{2,\mu}(\Omega) \cap C(\overline{\Omega})$, and*

$$\sup_\Omega d(x)^{\gamma-2}|u(x)| \le C \sup_\Omega d(x)^\gamma |f(x)|,$$

where C is a positive constant depending only on Ω and γ.

Remark 2. Notice that if $\sup_\Omega d(x)^\gamma |f(x)| < +\infty$, we also have $\sup_\Omega d(x)^{\gamma'} |f(x)| < +\infty$ for every $\gamma' > \gamma$. In particular, when $0 < \gamma \le 1$, Lemma 4 is still applicable by choosing some γ' such that $1 < \gamma' < 2$, but the estimate we obtain now is $|u(x)| \le Cd(x)^{2-\gamma'}$, $x \in \Omega$, which is not the optimal one.

We now apply this lemma to the problem (6). Since $0 \le a_k(x) \le a(x) \le C_2 d(x)^{-\gamma}$, and w_k^n are uniformly bounded independently of k, we have

$$\sup_\Omega d(x)^{\gamma-2}|w_k^n(x)| \le C \tag{7}$$

for some constant C not depending on k (this holds true in the whole range $0 < \gamma < 2$, cf. Remark 2). The uniform bounds for the sequence $\{w_k^n\}$ and a standard bootstrapping argument give local $C^{2,\mu}$ bounds, so that by means of a diagonal procedure we can obtain a convergent subsequence $w_k^n \to w_n$ in $C^2_{\rm loc}(\Omega)$ as $k \to +\infty$. Passing to the limit in (6) we obtain that $\Delta w_n = a(x)(w_n + n)^m$ in Ω, and $\sup_\Omega d(x)^{\gamma-2}|w_n(x)| \le C$ thanks to (7). Since $0 < \gamma < 2$, this implies $w_n \in C(\overline{\Omega})$ and $w_n = 0$ on $\partial\Omega$. The function $u_n = w_n + n$ is then the required solution to (P_n) (notice that u_n is positive, since $-n < w_n < 0$). By elliptic regularity, $u_n \in C^{2,\mu}(\Omega)$. Uniqueness and monotonicity with respect to n are a consequence of the maximum principle. □

The following lemma is the equivalent to Lemma 3 for $f(u) = e^u$. We omit the proof, which is essentially the same. The only point worth stressing is that we can find a subsolution to the truncated problem of the form $\underline{w} = A(|x|^2 - B)$ for sufficiently large A and B, and that uniform bounds for $e^{w_n^k}$ are enough to pass to the limit via Lemma 4.

Lemma 5. *Let $n \in \mathbb{N}$, and assume $a \in C^\mu(\Omega)$ verifies (A) with $0 < \gamma < 2$. Then the problem*

$$\begin{cases} \Delta v = a(x)e^v & in \quad \Omega \\ v = n & on \quad \partial\Omega \end{cases} \qquad (P'_n)$$

admits a unique solution $v_n \in C^{2,\mu}(\Omega) \cap C(\overline{\Omega})$. Moreover, v_n is increasing in n.

We finally prove that the sequences $\{u_n\}$ and $\{v_n\}$ constructed above converge, as $n \to +\infty$, to solutions of problems (P) and (P'), respectively.

Lemma 6. *Let $\{u_n\}$, $\{v_n\}$ be the sequences of solutions to problems (P_n) and (P'_n) given by Lemma 3 and Lemma 5, respectively. Then $u_n \to u$ in $C^2_{\rm loc}(\Omega)$, where $u \in C^{2,\mu}(\Omega)$ is a positive solution to (P) and $v_n \to v$ in $C^2_{\rm loc}(\Omega)$, where $v \in C^{2,\mu}(\Omega)$ is a solution to (P').*

Proof. Since a is strictly positive in Ω, there exists $a_0 > 0$ such that $a \ge a_0$. Then u_n verifies $\Delta u_n \ge a_0 u_n^m$ in Ω, and the maximum principle implies $u_n \le U$, where U is the unique solution to problem (P) with $a(x) \equiv a_0$ (cf. [3]). This gives local uniform bounds for u_n, and an argument similar to the used in the proof of Lemma 3 gives that $u_n \to u$ in $C^2_{\rm loc}(\Omega)$ for some function u. Passing to the limit in (P_n) we see that $\Delta u = a(x)u^m$, and since u_n is increasing it also follows that $u = +\infty$ on $\partial\Omega$. Finally, by elliptic regularity, $u \in C^{2,\mu}_{\rm loc}(\Omega)$, and is a classical solution to (P). The proof for the sequence $\{v_n\}$ is similar. □

3. Global estimates and uniqueness

This section is devoted to prove the global estimates (1) and (3), the nonexistence result for $\gamma \ge 2$ and the uniqueness of solutions to problems (P) and (P').

Theorem 7. *Assume $a \in C^\mu(\Omega)$ verifies hypothesis (A). Then there exist positive constants D_1, D_2 depending only on α and Ω, such that*

$$(D_1 \sup_\Omega d(x)^\gamma a(x))^{-\frac{1}{m-1}} d(x)^{-\alpha} \le u(x) \le (D_2 \inf_\Omega d(x)^\gamma a(x))^{-\frac{1}{m-1}} d(x)^{-\alpha} \quad in\ \Omega, \qquad (1)$$

for every positive solution u to (P).

Proof. Let u be a positive solution to (P). We show first that $\inf_\Omega d(x)^\alpha u(x) > 0$, $\sup_\Omega d(x)^\alpha u(x) < +\infty$.

Consider an arbitrary sequence $\{x_n\} \subset \Omega$ such that $d_n := d(x_n) \to 0$. In the ball of center x_n and radius $d_n/2$ we have $\Delta u \geq C_1(3/2)^{-\gamma} d_n^{-\gamma} u^m$. Setting

$$u_n(x) = \left(\frac{2^{\gamma-2}C_1}{3^\gamma}\right)^{\frac{1}{m-1}} d_n^\alpha\, u(x_n + \frac{d_n}{2}x), \quad x \in B(0,1)\ ,$$

we obtain $\Delta u_n \geq u_n^m$ in $B(0,1)$. By the maximum principle, $u_n(x) \leq U(x)$, $x \in B(0,1)$, where U stands for the unique positive solution to $\Delta U = U^m$ in $B(0,1)$, $U_{|\partial B(0,1)} = +\infty$. For $x = 0$ we obtain

$$d_n^\alpha u(x_n) \leq \left(\frac{3^\gamma}{2^{\gamma-2}C_1}\right)^{\frac{1}{m-1}} U(0)\ .$$

Since the sequence $\{x_n\}$ is arbitrary, this shows that $\sup_\Omega d(x)^\alpha u(x) < +\infty$.

To prove the lower bound, we let $v = u^{-p}$ with $0 < p < m-1$ to be chosen. Then

$$-\Delta v + \frac{p+1}{p}\frac{|\nabla v|^2}{v} = p\ a(x) v^{-r}\ ,$$

where $r = (m-1-p)/p > 0$. It follows that $-\Delta v \leq Cd(x)^{-2+p\alpha}$ in Ω (C will always denote a positive constant). We now choose p small enough so that $1 < 2 - p\alpha < 2$. Let ϕ be the unique solution to the problem $-\Delta\phi = Cd(x)^{-2+p\alpha}$, $\phi_{|\partial\Omega} = 0$, which exists due to Lemma 4. By the maximum principle and Lemma 4, $v \leq \phi \leq Cd(x)^{p\alpha}$, that is $u(x) \geq Cd(x)^{-\alpha}$ in Ω.

We finally prove (1). Set $p = 1 + 2/\alpha$, and let $\lambda > 0$ to be determined later. Since $m - p < 0$, we have

$$\begin{aligned} \Delta(\lambda u) &= \lambda^{1-p} d(x)^\gamma a(x)(d(x)^\alpha u(x))^{m-p}(\lambda u)^p \\ &\geq \lambda^{1-p} \inf\nolimits_\Omega d(x)^\gamma a(x)(\sup\nolimits_\Omega d(x)^\alpha u(x))^{m-p}(\lambda u)^p \\ &= (\lambda u)^p \quad \text{in } \Omega, \end{aligned}$$

if $\lambda = ((\inf_\Omega d(x)^\gamma a(x))(\sup_\Omega d(x)^\alpha u(x))^{m-p})^{\frac{1}{p-1}}$ (compare with [13]). Hence λu is a subsolution to the problem $\Delta U = U^p$ in Ω, $U_{|\partial\Omega} = +\infty$, which has a unique solution U. Since MU is a supersolution and $\lambda u \leq MU$ in Ω for large M (observe that $U \sim Cd^{-\alpha}$ near $\partial\Omega$, so that the inequality can be achieved), the method of sub and supersolutions (cf. Lemma 4 in [14] and Lemma 2 in [13]) yields $\lambda u \leq U$. Note that the maximum principle could not be used directly, since we do not know a priori the relation between λu and U near the boundary. A little algebra now shows that this leads indeed to the upper estimate in (1). The lower estimate is proved in a similar way. □

The following theorem is the analogue of Theorem 7 in the exponential case. The proof is essentially the same and will be omitted.

Theorem 8. *Assume $a \in C^\mu(\Omega)$ verifies hypothesis (A). Then there exist positive constants D_1', D_2' depending only on α and Ω, such that*

$$D_1'(2-\gamma)(\sup_\Omega d(x)^\gamma a(x))^{-1} d(x)^{\gamma-2} \leq e^{v(x)} \leq D_2'(2-\gamma)(\inf_\Omega d(x)^\gamma a(x))^{-1} d(x)^{\gamma-2} \quad \text{in } \Omega, \quad (3)$$

for every solution v to (P').

For $\gamma \geq 2$ the estimates (1) and (3) imply that all solutions are bounded in $\overline{\Omega}$. We thus have the following corollary. See Lemma 5.1 in [16] for a related result.

Corollary 9. *Assume $a \in C^\mu(\Omega)$ verifies hypothesis (A) and $\gamma \geq 2$. Then problems (P) and (P') have no solutions.*

We now study the uniqueness of solutions of problems (P) and (P'). Observe that (1) and (3) are not enough to conclude that the quotients of two positive solutions to (P) tend to 1 at $\partial\Omega$, which has been often used to prove uniqueness. Instead we adapt the argument in Theorem 3.4 of [19].

Theorem 10. *Assume $a \in C^\mu(\Omega)$ verifies (A) and $0 < \gamma < 2$. Then problem (P) has a unique positive solution.*

Proof. Let u, v be positive solutions to (P), and assume there exist $x_0 \in \Omega$ and $k > 1$ such that $u(x_0) > kv(x_0)$. Let $\Omega_0 := \{u > kv\} \cap B_r(x_0)$, where $r = d(x_0)/2$. By (1) we have

$$\Delta(u - kv) > a(x)(k^{m-1} - 1)kv^m \geq \kappa_1 r^{-\alpha-2} k$$

in Ω_0, where κ_1 is some positive constant. Set $w(x) = (\kappa_1 r^{-\alpha-2} k)(r^2 - |x - x_0|^2)/2N$. Then $\Delta(u - kv + w) \geq 0$ in Ω_0, and from the maximum principle we deduce the existence of $x_1 \in \partial\Omega_0$ such that

$$u(x_0) - kv(x_0) + w(x_0) \leq u(x_1) - kv(x_1) + w(x_1)\ .$$

It is easy to see that $x_1 \in \partial B_r(x_0)$. Then we obtain that $(\kappa_1 r^{-\alpha} k)/2N \leq u(x_1) - kv(x_1)$, which together with the upper estimate for v, leads to the existence of $\kappa_2 > 0$ such that $u(x_1) > (1 + \kappa_2)kv(x_1)$. Proceeding inductively, we find a sequence $\{x_n\} \subset \Omega$ such that $u(x_n) > (1 + \kappa_2)^n kv(x_n)$, which contradicts the fact that the quotients of any two positive solutions must be bounded. Hence, $u \leq v$, and interchanging the roles of u and v we obtain $u \equiv v$. This proves the theorem. □

Theorem 11. *Assume $a \in C^\mu(\Omega)$ verifies (A) and $0 < \gamma < 2$. Then problem (P') has a unique solution.*

Proof. The argument is similar to the one used in Theorem 10. Let u, v be solutions to (P') and notice that (3) implies that $u - v$ is bounded. Assume there exist $x_0 \in \Omega$ and $k > 0$ such that $u > v + k$. Define $\Omega_0 := \{u > v + k\} \cap B_r(x_0)$, for $r = d(x_0)/2$. Then

$$\Delta(u - v - k) > a(x)e^v(e^k - 1) \geq \kappa_1 r^{-2}$$

in Ω_0. Setting $w(x) = (\kappa_1 r^{-2})(r^2 - |x - x_0|^2)/2N$, we obtain as before the existence of $x_1 \in \partial B_r(x_0)$ such that $w(x_0) < u(x_1) - v(x_1) - k$. In particular, $u(x_1) > v(x_1) + k + \kappa_1/2N$. Iterating this procedure, we obtain a sequence $\{x_n\} \subset \Omega$ such that $u(x_n) > v(x_n) + k + n\kappa_2/2N$, which contradicts the boundedness of $u - v$. Thus, $u \leq v$ and similarly $u \geq v$, which proves uniqueness. □

Remarks 3. a) For problem (P'), we can use a standard procedure to show uniqueness, since the global estimate (3) shows that the quotient of two arbitrary solutions tends to one as we approach the boundary of Ω (cf. Remarks 1 b)). Indeed, let u, v be solutions to (P'). By replacing u, v by $u+1$, $v+1$ and $a(x)$ by $a(x)e^{-1}$, we can assume that $u, v \geq 1$. Let $w = u/v$. Then

$$v\Delta w + 2\nabla v \nabla w = a(x)e^{v}(e^{(w-1)v} - w) \quad \text{in } \Omega.$$

Set $\Omega^+ := \{x \in \Omega : \ w(x) > 1\}$, and assume $\Omega^+ \neq \emptyset$ (observe that $w = 1$ on $\partial\Omega$, and this implies $w = 1$ on $\partial\Omega^+$). Then since the function $h(\tau) = e^{(\tau-1)v} - \tau$ is positive for $\tau \geq 1$, we have $v\Delta w + 2\nabla v \nabla w \geq 0$ in Ω^+, and by the maximum principle $w \leq 1$ in Ω^+, a contradiction. Thus $\Omega^+ = \emptyset$, i.e. $w \leq 1$ in Ω, and the symmetric argument gives $w \equiv 1$, which shows uniqueness.

b) The proof of Theorems 10 and 11 is also valid for *unbounded* domains, as long as we have estimates like (1) and (3) for all solutions (this will be used in Section 4).

4. Estimates near the boundary

In this final section, we are obtaining estimates near the boundary both for solutions to (P) and (P') and their normal derivatives. Our proof is based on a rescaling argument. We will assume that $a \in C^{\mu}(\Omega)$ verifies hypotheses (A) for $0 < \gamma < 2$ and (A').

We are performing all the calculations for problem (P), since the translation of the argument to problem (P') is straightforward.

Proof of estimates (2). Let $x_0 \in \partial\Omega$, and $\{x_n\} \subset \Omega$ such that $x_n \to x_0$. Choose an open neighbourhood $\mathcal{U}$ of x_0 so that $\partial\Omega$ admits $C^{2,\mu}$ local coordinates $\xi : \mathcal{U} \to \mathbb{R}^N$, and $x \in \mathcal{U} \cap \Omega$ if and only if $\xi_1(x) > 0$ ($\xi = (\xi_1, \xi_2, \dots, \xi_N)$). We can moreover assume $\xi(x_0) = 0$. If $u(x) = \bar{u}(\xi(x))$, then we have the equation

$$\sum_{i,j=1}^{N} a_{ij}(\xi) \frac{\partial^2 \bar{u}}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{N} b_i(\xi) \frac{\partial \bar{u}}{\partial \xi_i} = \bar{a}(\xi)\bar{u}^m$$

in $\xi(\mathcal{U} \cap \Omega)$, where $a(x) = \bar{a}(\xi(x))$, a_{ij}, b_i are (at least) C^{μ}, and $a_{ij}(0) = \delta_{ij}$. Denote by η_n the projections onto $\xi(\mathcal{U} \cap \partial\Omega)$ of $\xi(x_n)$, and introduce the functions

$$u_n(y) = d_n^{\alpha}\, \bar{u}(\eta_n + d_n y)\ ,$$

where $d_n = d(\xi(x_n))$. Notice that $\xi(x_n) = \eta_n + d_n(1, 0, \dots, 0)$. The function u_n satisfies the equation

$$\sum_{i,j=1}^{N} a_{ij}(\eta_n + d_n y) \frac{\partial u_n}{\partial \xi_i \partial \xi_j} + d_n \sum_{i=1}^{N} b_i(\eta_n + d_n y) \frac{\partial u_n}{\partial \xi_i} = d_n^{\gamma} \bar{a}(\eta_n + d_n y) u_n^m.$$

On the other hand, estimates (1) imply that, for y in compact subsets K of $D := \{y \in \mathbb{R}^N : \ y_1 > 0\}$, there exists $n_0 = n_0(K)$ such that $C y_1^{-\alpha} \leq u_n(y) \leq C' y_1^{-\alpha}$ for $n \geq n_0$, where C, C' are positive constants not depending on K. These estimates, together with the equation, a bootstrap argument and a diagonal procedure, allow us to obtain a subsequence (still labelled by $\{u_n\}$) such that $u_n \to u_0$ in $C^2_{\text{loc}}(D)$. In particular, we obtain that

$$\begin{cases} \Delta u_0 = C_0(x_0) y_1^{-\gamma} u_0^m & \text{in} \quad D \\ C y_1^{-\alpha} \le u_0 \le C' y_1^{-\alpha} . \end{cases}$$

Thanks to Theorem 10 and Remarks 3 b), we see that this problem has a unique positive solution, which can be checked to be

$$u_0(y) = \left(\frac{\alpha(\alpha+1)}{C_0(x_0)} \right)^{\frac{1}{m-1}} y_1^{-\alpha} .$$

Thus, taking $y = (1, 0, \dots, 0)$, we arrive at

$$d_n^\alpha u(x_n) \to \left(\frac{\alpha(\alpha+1)}{C_0(x_0)} \right)^{\frac{1}{m-1}}, \quad d_n^{\alpha+1} \frac{\partial u}{\partial \xi_1}(x_n) \to -\alpha \left(\frac{\alpha(\alpha+1)}{C_0(x_0)} \right)^{\frac{1}{m-1}},$$
$$d_n^{\alpha+2} \frac{\partial^2 u}{\partial \xi_1^2}(x_n) \to \alpha(\alpha+1) \left(\frac{\alpha(\alpha+1)}{C_0(x_0)} \right)^{\frac{1}{m-1}} .$$

This proves (2), since the sequence $\{x_n\}$ is arbitrary.

Finally notice that if $a(x) \in C^{k,\mu}(\Omega)$ and Ω is $C^{k+2,\mu}$, for some $k \in \mathbb{N}$ (Remarks 1 d)), we obtain $u_n \to u_0$ in $C^{k+2}_{\text{loc}}(\Omega)$, so we can establish estimates for the derivatives up to order $k+2$. □

Remark 4. In case the function $C_0(x_0)$ is constant on $\partial\Omega$, there is another way of producing the first estimate in (2) for problem (P). Indeed, setting $\Omega_\delta := \{x \in \Omega : \ d(x) < \delta\}$ for positive small δ and $p = 1 + 2/\alpha$, we have

$$\Delta(\lambda u) \le \lambda^{1-p} \sup_{\Omega_\delta} (d(x)^\gamma a(x)) (\inf d(x)^\alpha u(x))^{m-p} (\lambda u)^p$$

in Ω_δ. Choosing $\lambda = \lambda(\delta) := (\sup_{\Omega_\delta} d(x)^\gamma a(x) (\inf_{\Omega_\delta} d(x)^\alpha u(x))^{m-p})^{\frac{1}{p-1}}$, we obtain $\Delta(\lambda u) \le (\lambda u)^p$ in Ω_δ. A similar calculation as in [8] shows then that for a suitable constant $C > 0$, $\lambda u + C\delta^{-\alpha} \ge U$ in Ω_δ, where U stands for the unique positive solution to $\Delta U = U^p$ in Ω, $U = +\infty$ on $\partial\Omega$. After some manipulations, this leads to

$$\liminf_{d \to 0} d(x)^\alpha u(x) \ge \left(\frac{\alpha(\alpha+1)}{C_0} \right)^{\frac{1}{m-1}} ,$$

and a similar reasoning proves the other inequality.

As can be checked, this proof can not be adapted to nonconstant C_0, since it is global in nature, and does not provide any information on the growth of the normal derivatives.

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